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Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(N))$ via contraction method and mapping

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Abstract

Using a contraction procedure developed earlier, we construct, in the first part of the present paper, the Jordanian quantum Hopf algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ which has a *remarkably simple coalgebraic structure* and contains the Jordanian algebra $\mathcal{U}_{\mathsf{h}}(sl(2))$, obtained by Ohn, as a Hopf subalgebra. A nonlinear map between the quantum $\mathcal{U}_{\mathsf{h}}(sl(3))$ and the classical $\mathcal{U}(sl(3))$ algebras is then established; and the universal $\mathcal{R}_{\mathsf{h}}$-matrix of the $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra is given. In the second part, we give the higher dimensional Jordanian algebras $\mathcal{U}_{\mathsf{h}}(sl(N))$ for all N. The universal $\mathcal{R}_{\mathsf{h}}$-matrix of the $\mathcal{U}_{\mathsf{h}}(sl(N))$ algebra is also given.

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1. Introduction

It is well known that the enveloping algebra $\mathcal{U}(sl(N))$ of the Lie algebra $sl(N)$ has two quantizations: the first one called the *Drinfeld–Jimbo deformation* or the *standard quantum deformation* [1, 2] is quasitriangular, whereas the second one called the *Jordanian deformation* or the *non-standard quantum deformation* [3] is triangular ($\mathcal{R}_{21}\mathcal{R} = I$). A typical example of Jordanian quantum algebras was first introduced by Ohn [4]. In general, non-standard quantum algebras are obtained by applying Drinfeld twist [5] to the corresponding Lie algebras. A twisting that produces an algebra isomorphic to the Ohn algebra [4] is found in [6, 7].

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Recently, the twisting procedure was extensively employed to study a wide variety of Jordanian deformed algebras, such as $\mathcal{U}_{\mathsf{h}}(sl(N))$ algebra [8–12], symplectic algebra $\mathcal{U}_{\mathsf{h}}(sp(N))$ [13], orthogonal algebra $\mathcal{U}_{\mathsf{h}}(so(N))$ [14–17] and orthosymplectic superalgebra $\mathcal{U}_{\mathsf{h}}(osp(1|2))$ [18, 19]. It follows from these studies that:

1. the non-standard quantum algebras have undeformed commutation relations;
2. the Jordanian deformation appears only in the coalgebraic structure;
3. the coproduct and the antipode maps have very complicated forms in comparison with the Drinfeld–Jimbo and the Ohn deformations.

So far Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(N))$ has been explicitly written, with a simple coalgebra but with deformed commutation relations, only for $N = 2$ [4]. This amounts to a choice of an appropriate basis, in which the commutation relations are deformed but the corresponding coalgebraic structure remains simple. Following this approach we here construct the Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$, wherein we use the contraction procedure developed in [20, 21] and an analogue of the map introduced before [21, 22]. The $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra presented here has the following properties:

1. the Ohn algebra $\mathcal{U}_{\mathsf{h}}(sl(2))$ is included in our $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebraic structure in a natural way as a Hopf subalgebra and arises here from the generators associated with the highest root;
2. our Jordanian deformed $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra may be regarded as the dual Hopf algebra of the function algebra $Fun_{\mathsf{h}}(SL(3))$ studied in [23];
3. with our choice of the basis the present $\mathcal{U}_{\mathsf{h}}(sl(3))$ Hopf algebra has *deformed commutation relations*; but is endowed with *a relatively simpler coalgebraic structure compared to those in the previous studies* [8–11]. Contrasting these previous papers, this *simplicity of the present coalgebraic structure* is a distinguishing feature of our study. Pertinent to the *full* Hopf structure of the $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra, we obtain its universal $\mathcal{R}_{\mathsf{h}}$-matrix comprising the generators associated with the highest root.

Implementing our contraction technique we subsequently obtain higher dimensional Jordanian quantum algebras $\mathcal{U}_{\mathsf{h}}(sl(N))$ for arbitrary values of *N*. Here also our *commutation relations are deformed*, and our *coalgebraic structures are considerably simpler than those found elsewhere* [8–11]. Generalizing our result on the $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra, we obtain the universal $\mathcal{R}_{\mathsf{h}}$-matrix of the $\mathcal{U}_{\mathsf{h}}(sl(N))$ algebra.

The manuscript is organized as follows: the Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ is introduced via a nonlinear map and proved to be a Hopf algebra endowed with a triangular universal $\mathcal{R}_{\mathsf{h}}$-matrix in section 2. The irreducible representations (irreps.) of the $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra are also given. Higher dimensional algebras $\mathcal{U}_{\mathsf{h}}(sl(N))$, $N \geqslant 4$, are presented in sections 3 and 4. We conclude in section 5.

2. $\mathcal{U}_{\mathsf{h}}(sl(3))$: map, Hopf algebra, irreps and $\mathcal{R}_{\mathsf{h}}$-matrix

For our purpose, the deformation parameter h is an arbitrary complex number. It was proved in [21] that the $\mathcal{R}_{\mathsf{h}}$-matrix of the Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ can be obtained from the $\mathcal{R}_q$-matrix associated with the Drinfeld–Jimbo quantum algebra $\mathcal{U}_q(sl(3))$ through a specific contraction which is singular in the $q \to 1$ limit. For the transformed matrix, the singularities, however, cancel yielding a well-defined construction. For the sake of completeness here we briefly describe the well-known [24] Hopf structure of the $\mathcal{U}_q(sl(3))$ algebra.

Choosing the Chevalley generators corresponding to the simple roots of the $\mathcal{U}_q(sl(3))$ algebra as $(\hat{e}_i, \hat{f}_i(=\hat{e}_{-i}), h_i \,|\, i = (1, 2))$, we define $\hat{e}_3 = \hat{e}_1\hat{e}_2 - q^{-1}\hat{e}_2\hat{e}_1$, $\hat{f}_3 = \hat{f}_2\hat{f}_1 -$

$q\hat{f}_1\hat{f}_2$, $h_3 = h_1 + h_2$. The Hopf structure of the $\mathcal{U}_q(sl(3))$ algebra is given by

$$\begin{aligned}
&[h_i, h_j] = 0 \qquad [h_i, \hat{e}_{\pm j}] = \pm a_{ij}\hat{e}_{\pm j} \qquad [\hat{e}_i, \hat{e}_{-j}] = \delta_{ij}[h_i] \\
&\hat{e}_1\hat{e}_3 = q\hat{e}_3\hat{e}_1 \qquad \hat{e}_2\hat{e}_3 = q^{-1}\hat{e}_3\hat{e}_2 \qquad \hat{f}_1\hat{f}_3 = q\hat{f}_3\hat{f}_1 \qquad \hat{f}_2\hat{f}_3 = q^{-1}\hat{f}_3\hat{f}_2 \\
&\Delta_q(h_i) = h_i \otimes 1 + 1 \otimes h_i \qquad \Delta_q(\hat{e}_{\pm i}) = \hat{e}_{\pm i} \otimes q^{h_i/2} + q^{-h_i/2} \otimes \hat{e}_{\pm i} \\
&\epsilon_q(h_i) = \epsilon_q(\hat{e}_{\pm i}) = 0 \qquad S_q(h_i) = -h_i \qquad S_q(\hat{e}_{\pm i})) = -q^{\pm 1}\hat{e}_{\pm i}
\end{aligned} \tag{1}$$

where $[\mathcal{X}] = \frac{q^{\mathcal{X}} - q^{-\mathcal{X}}}{q - q^{-1}}$. The Cartan matrix for the $sl(3)$ algebra reads $a = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The universal $\mathcal{R}_q$-matrix of the $\mathcal{U}_q(sl(3))$ algebra is given by

$$\begin{aligned}
\mathcal{R}_q = {} & q^{\sum_{i,j}(a^{-1})_{ij}h_i \otimes h_j} \exp_{q^{-2}}\left(\lambda\hat{e}_2 q^{h_2/2} \otimes q^{-h_2/2}\hat{f}_2\right) \exp_{q^{-2}}\left(\lambda\hat{e}_3 q^{h_3/2} \otimes q^{-h_3/2}\hat{f}_3\right) \\
& \times \exp_{q^{-2}}\left(\lambda\hat{e}_1 q^{h_1/2} \otimes q^{-h_1/2}\hat{f}_1\right)
\end{aligned} \tag{2}$$

where $\lambda = q - q^{-1}$, $\exp_q(\mathcal{X}) = \sum_{n=0}^{\infty} \mathcal{X}^n/\{n\}_q!$, $\{n\}_q! = \{n\}_q\{n-1\}_q!$, $\{0\}_q!, = 1$ and $\{n\}_q = (1 - q^n)/(1 - q)$. We subsequently denote the classical $(q = 1)$ generators of the $sl(3)$ algebra by $h_1, h_2, h_3 = h_1 + h_2, e_1, e_2, e_3 = e_1e_2 - e_2e_1, f_1, f_2$ and $f_3 = f_2f_1 - f_1f_2$.

Although the present contraction method is generic in character and may be used to extract the Jordanian R_{h}-matrix for arbitrary representations in the two tensor product sectors, we, for brevity and simplicity, limit ourselves to (fundamental irrep) $\otimes$ (arbitrary irrep). The R_q-matrix of the $\mathcal{U}_q(sl(3))$ algebra in the representation (fund.) $\otimes$ (arb.) reads

$$\begin{aligned}
R_q &= \left(\pi_{(\text{fund.})} \otimes \pi_{(\text{arb.})}\right)\mathcal{R}_q \\
&= \begin{pmatrix}
q^{\frac{1}{3}(2h_1+h_2)} & q^{\frac{1}{3}(2h_1+h_2)}\Lambda_{12} & q^{\frac{1}{3}(2h_1+h_2)}\Lambda_{13} \\
0 & q^{-\frac{1}{3}(h_1-h_2)} & q^{-\frac{1}{3}(h_1-h_2)}\Lambda_{23} \\
0 & 0 & q^{-\frac{1}{3}(h_1+2h_2)}
\end{pmatrix}
\end{aligned} \tag{3}$$

where

$$\Lambda_{12} = q^{-1/2}\lambda q^{-h_1/2}\hat{f}_1 \qquad \Lambda_{13} = q^{-1/2}\lambda q^{-h_3/2}\hat{f}_3 \qquad \Lambda_{23} = q^{-1/2}\lambda q^{-h_2/2}\hat{f}_2. \tag{4}$$

We have shown in [21] that the non-standard R_{h}-matrix in the (fund.) $\otimes$ (arb.) representation arises from the corresponding R_q-matrix as follows:

$$\begin{aligned}
R_{\mathsf{h}} = {} & \lim_{q\to 1}\left[E_q\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right)_{(\text{fund.})} \otimes E_q\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right)_{(\text{arb.})}\right]^{-1} \\
& \times R_q\left[E_q\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right)_{(\text{fund.})} \otimes E_q\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right)_{(\text{arb.})}\right] \\
= {} & \lim_{q\to 1}\begin{pmatrix}
E_q^{-1}\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right) & 0 & -\frac{\mathsf{h}}{q-1}E_q^{-1}\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right) \\
0 & E_q^{-1}\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right) & 0 \\
0 & 0 & E_q^{-1}\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right)
\end{pmatrix} \\
& \times R_q\begin{pmatrix}
E_q\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right) & 0 & \frac{\mathsf{h}}{q-1}E_q\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right) \\
0 & E_q\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right) & 0 \\
0 & 0 & E_q\left(\frac{\mathsf{h}\hat{e}_3}{q-1}\right)
\end{pmatrix} \\
= {} & \begin{pmatrix}
T & 2\mathsf{h}T^{-1/2}e_2 & -\frac{\mathsf{h}}{2}(T + T^{-1})(h_1 + h_2) + \frac{\mathsf{h}}{2}(T - T^{-1}) \\
0 & I & -2\mathsf{h}T^{1/2}e_1 \\
0 & 0 & T^{-1}
\end{pmatrix}
\end{aligned} \tag{5}$$

where

$$T = \mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2} \qquad T^{-1} = -\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}. \tag{6}$$

The deformed exponential in (5) is defined by

$$E_q(\mathcal{X}) = \sum_{n=0}^{\infty} \frac{\mathcal{X}^n}{[n]!} \qquad \text{where} \quad [n]! = [n] \times [n-1]! \quad [0]! = 1. \tag{7}$$

The following properties can be pointed out:

1. The corner elements of (5) have exactly the same structure as in the R_{h}-matrix of the $\mathcal{U}_{\mathsf{h}}(sl(2))$ algebra. This indicates that the classical generators e_3, $h_3 = h_1 + h_2$ and f_3 of the $\mathcal{U}(sl(3))$ algebra are deformed (for the non-standard quantization: $\mathcal{U}(sl(3)) \to \mathcal{U}_{\mathsf{h}}(sl(3))$) as follows [21, 22]:

$$\begin{aligned} &T = \mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2} \qquad T^{-1} = -\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2} \\ &H_3 = \sqrt{1+\mathsf{h}^2e_3^2}\,h_3 \qquad F_3 = f_3 - \frac{\mathsf{h}^2}{4} e_3\left(h_3^2 - 1\right) \end{aligned} \tag{8}$$

and the deformed generators evidently satisfy the commutation relations [4]

$$\begin{aligned} &TT^{-1} = T^{-1}T = 1 \qquad [H_3, T] = T^2 - 1 \qquad [H_3, T^{-1}] = T^{-2} - 1 \\ &[T, F_3] = \frac{\mathsf{h}}{2}(H_3T + TH_3) \qquad \left[T^{-1}, F_3\right] = -\frac{\mathsf{h}}{2}\left(H_3T^{-1} + T^{-1}H_3\right) \\ &[H_3, F_3] = -\tfrac{1}{2}\left(TF_3 + F_3T + T^{-1}F_3 + F_3T^{-1}\right). \end{aligned} \tag{9}$$

On defining

$$E_3 = \mathsf{h}^{-1} \ln T = \mathsf{h}^{-1} \operatorname{arcsinh} \mathsf{h}e_3 \tag{10}$$

it follows that the elements (H_3, E_3, F_3) satisfy the relations of the $\mathcal{U}_{\mathsf{h}}(sl(2))$ algebra [4]

$$\begin{aligned} &[H_3, E_3] = 2\frac{\sinh \mathsf{h}E_3}{\mathsf{h}} \\ &[H_3, F_3] = -F_3\left(\cosh \mathsf{h}E_3\right) - \left(\cosh \mathsf{h}E_3\right)F_3 \\ &[E_3, F_3] = H_3 \end{aligned} \tag{11}$$

where it is obvious that as $\mathsf{h} \to 0$, we have $(H_3, E_3, F_3) \to (h_3, e_3, f_3)$. The algebraic property (11) makes the embedding $\mathcal{U}_{\mathsf{h}}(sl(2)) \subset \mathcal{U}_{\mathsf{h}}(sl(3))$ evident.

2. Expression (5) of the R_{h}-matrix indicates that the simple root generators e_1 and e_2 are deformed as follows:

$$E_1 = \sqrt{\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}}\,e_1 = T^{1/2}e_1 \qquad E_2 = \sqrt{\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}}\,e_2 = T^{1/2}e_2. \tag{12}$$

To complete our $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra, we introduce the following h-deformed generators:

$$\begin{aligned} &F_1 = \sqrt{-\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}}\,f_1 + \frac{\mathsf{h}}{2}\sqrt{\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}}\,e_2h_3 = T^{-1/2}\left(f_1 + \frac{\mathsf{h}}{2}e_2Th_3\right) \\ &F_2 = \sqrt{-\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}}\,f_2 - \frac{\mathsf{h}}{2}\sqrt{\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}}\,e_1h_3 = T^{-1/2}\left(f_2 - \frac{\mathsf{h}}{2}e_1Th_3\right) \\ &H_1 = \left(-\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}\right)\left(\sqrt{1+\mathsf{h}^2e_3^2}\,h_1 + \frac{\mathsf{h}}{2}e_3(h_1 - h_2)\right) = h_1 - \frac{\mathsf{h}}{2}e_3T^{-1}h_3 \\ &H_2 = \left(-\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}\right)\left(\sqrt{1+\mathsf{h}^2e_3^2}\,h_2 - \frac{\mathsf{h}}{2}e_3(h_1 - h_2)\right) = h_2 - \frac{\mathsf{h}}{2}e_3T^{-1}h_3. \end{aligned} \tag{13}$$

Expressions (8), (12) and (13) constitute a realization of the Jordanian algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ with the classical generators via a nonlinear map. This immediately yields the irreducible representations (irreps.) of $\mathcal{U}_{\mathsf{h}}(sl(3))$ in an explicit and simple manner.

Proposition 1. *The Jordanian algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ is an associative algebra over $\mathbb{C}$ generated by $H_1, H_2, H_3, E_1, E_2, T, T^{-1}, F_1, F_2$ and F_3, satisfying, along with (9), the commutation relations*

$$
\begin{aligned}
&[H_1, H_2] = 0 \qquad \left[H_1, T^{-1}H_3\right] = \left[H_2, T^{-1}H_3\right] = 0 \qquad [H_1, E_1] = 2E_1 \\
&[H_2, E_2] = 2E_2 \qquad [H_1, E_2] = -E_2 \qquad [H_2, E_1] = -E_1 \\
&\left[T^{-1}H_3, E_1\right] = E_1 \qquad \left[T^{-1}H_3, E_2\right] = E_2 \qquad [H_1, F_1] = -2F_1 + \mathsf{h}E_2T^{-1}H_3 \\
&[H_2, F_2] = -2F_2 - \mathsf{h}E_1T^{-1}H_3 \qquad [H_1, F_2] = F_2 - \mathsf{h}E_1T^{-1}H_3 \\
&[H_2, F_1] = F_1 + \mathsf{h}E_2T^{-1}H_3 \qquad [TH_3, F_1] = -T^2F_1 \qquad [TH_3, F_2] = -T^2F_2 \\
&\left[T^{-1}E_1, F_1\right] = \tfrac{1}{2}(T + T^{-1})H_1 + \tfrac{1}{2}(T - T^{-1})H_2 \\
&\left[T^{-1}E_2, F_2\right] = \tfrac{1}{2}(T + T^{-1})H_2 + \tfrac{1}{2}(T - T^{-1})H_1 \\
&\left[T^{-1}E_1, F_2\right] = 0 \qquad \left[T^{-1}E_2, F_1\right] = 0 \qquad [E_1, E_2] = \frac{1}{2\mathsf{h}}(T^2 - 1) \\
&[TF_2, TF_1] = T\left(F_3 - \frac{\mathsf{h}}{2}H_3TH_3 - \frac{\mathsf{h}}{8}(T - T^{-1})\right) \qquad [TH_1, T] = \tfrac{1}{2}(T^2 - 1) \\
&\left[TH_1, T^{-1}\right] = \tfrac{1}{2}(T^{-2} - 1) \qquad [TH_2, T] = \tfrac{1}{2}(T^2 - 1) \qquad \left[TH_2, T^{-1}\right] = \tfrac{1}{2}(T^{-2} - 1) \\
&[H_1, F_3] = -\frac{T^{-1}}{4}\left(TF_3 + F_3T + T^{-1}F_3 + F_3T^{-1}\right) - \frac{\mathsf{h}}{4}T^{-1}H_3^2 - \frac{\mathsf{h}}{4}H_3T^{-1}H_3 \\
&[H_2, F_3] = -\frac{T^{-1}}{4}\left(TF_3 + F_3T + T^{-1}F_3 + F_3T^{-1}\right) - \frac{\mathsf{h}}{4}T^{-1}H_3^2 - \frac{\mathsf{h}}{4}H_3T^{-1}H_3 \\
&[E_1, T] = \left[E_1, T^{-1}\right] = [E_2, T] = [E_2, T^{-1}] = 0 \qquad [F_1, T] = \mathsf{h}TE_2 \\
&\left[F_1, T^{-1}\right] = -\mathsf{h}T^{-1}E_2 \qquad [F_2, T] = -\mathsf{h}TE_1 \qquad \left[F_2, T^{-1}\right] = \mathsf{h}T^{-1}E_1 \\
&[E_1, F_3] = -\tfrac{1}{2}(TF_2 + F_2T) \qquad [E_2, F_3] = \tfrac{1}{2}(TF_1 + F_1T) \\
&[F_1, F_3] = \mathsf{h}TF_1 - \mathsf{h}E_2F_3 + \frac{\mathsf{h}^2}{4}TE_2 \qquad [F_2, F_3] = \mathsf{h}TF_2 + \mathsf{h}E_1F_3 - \frac{\mathsf{h}^2}{4}TE_1.
\end{aligned}
\tag{14}
$$

Here we have quoted only the final results. To obtain the realizations of H_1 and H_2 given in (13), we, in analogy with (8), started with the ansatz $\sqrt{1 + \mathsf{h}^2e_3^2}h_1$ and $\sqrt{1 + \mathsf{h}^2e_3^2}h_2$ for these generators, respectively. It is easy to see that

$$
\begin{aligned}
\left[\sqrt{1 + \mathsf{h}^2e_3^2}h_1, F_3\right] &= -\frac{1}{4}\left(TF_3 + F_3T + T^{-1}F_3 + F_3T^{-1}\right) \\
&\quad + \frac{\mathsf{h}^2}{4}(e_3(h_1 - h_2)H_3 + H_3e_3(h_1 - h_2)) \\
\left[\sqrt{1 + \mathsf{h}^2e_3^2}h_2, F_3\right] &= -\frac{1}{4}\left(TF_3 + F_3T + T^{-1}F_3 + F_3T^{-1}\right) \\
&\quad - \frac{\mathsf{h}^2}{4}(e_3(h_1 - h_2)H_3 + H_3e_3(h_1 - h_2)).
\end{aligned}
\tag{15}
$$

Then, if we add to $\sqrt{1+\mathsf{h}^2 e_3^2}h_1$ and deduct from $\sqrt{1+\mathsf{h}^2 e_3^2}h_2$ the term $\frac{\mathsf{h}}{2}e_3(h_1-h_2)$, we obtain

$$\begin{aligned}
\left[\left(\sqrt{1+\mathsf{h}^2 e_3^2}h_1+\frac{\mathsf{h}}{2}e_3(h_1-h_2)\right), F_3\right] &= -\frac{1}{4}\left(TF_3+F_3T+T^{-1}F_3+F_3T^{-1}\right)\\
&\quad +\frac{\mathsf{h}}{4}T(h_1-h_2)H_3+\frac{\mathsf{h}}{4}H_3T(h_1-h_2)\\
\left[\left(\sqrt{1+\mathsf{h}^2 e_3^2}h_2-\frac{\mathsf{h}}{2}e_3(h_1-h_2)\right), F_3\right] &= -\frac{1}{4}\left(TF_3+F_3T+T^{-1}F_3+F_3T^{-1}\right)\\
&\quad -\frac{\mathsf{h}}{4}T(h_1-h_2)H_3-\frac{\mathsf{h}}{4}H_3T(h_1-h_2).
\end{aligned} \tag{16}$$

These commutation relations suggest the realizations $H_1 \sim \left(\sqrt{1+\mathsf{h}^2 e_3^2}h_1+\frac{\mathsf{h}}{2}e_3(h_1-h_2)\right)$ and $H_2 \sim \left(\sqrt{1+\mathsf{h}^2 e_3^2}h_2-\frac{\mathsf{h}}{2}e_3(h_1-h_2)\right)$. Finally, to preserve the Cartan subalgebra, we are obliged to multiply both of these expressions by T^{-1}. The resultant maps for H_1 and H_2 are quoted in (13). The expressions of F_1 and F_2 are obtained in a similar way. Expressions (8), (12) and (13) may be looked at now as a particular realization of the $\mathcal{U}_{\mathsf{h}}(sl(3))$ generators. Other invertible maps relating the Jordanian and the classical generators may also be considered.

Proposition 2. *In terms of the Chevalley generators* $\{E_1, E_2, F_1, F_2, H_1, H_2\}$, *the Jordanian algebra* $\mathcal{U}_{\mathsf{h}}(sl(3))$ *is defined as follows:*

$$\begin{aligned}
&T=(1+2\mathsf{h}[E_1,E_2])^{1/2} \qquad T^{-1}=(1+2\mathsf{h}[E_1,E_2])^{-1/2} \qquad [H_1,H_2]=0\\
&[H_1,E_1]=2E_1 \qquad [H_2,E_2]=2E_2 \qquad [H_1,E_2]=-E_2 \qquad [H_2,E_1]=-E_1\\
&[H_1,F_1]=-2F_1+\mathsf{h}E_2(H_1+H_2) \qquad [H_2,F_2]=-2F_2-\mathsf{h}E_1(H_1+H_2)\\
&[H_1,F_2]=F_2-\mathsf{h}E_1(H_1+H_2) \qquad [H_2,F_1]=F_1+\mathsf{h}E_2(H_1+H_2)\\
&\left[T^{-1}E_1,F_1\right]=\tfrac{1}{2}(T+T^{-1})H_1+\tfrac{1}{2}(T-T^{-1})H_2\\
&\left[T^{-1}E_2,F_2\right]=\tfrac{1}{2}(T+T^{-1})H_2+\tfrac{1}{2}(T-T^{-1})H_1 \qquad \left[T^{-1}E_1,F_2\right]=\left[T^{-1}E_2,F_1\right]=0\\
&E_1^2E_2-2E_1E_2E_1+E_2E_1^2=0 \qquad E_2^2E_1-2E_2E_1E_2+E_1E_2^2=0\\
&(TF_1)^2TF_2-2TF_1TF_2TF_1+TF_2(TF_1)^2=0\\
&(TF_2)^2TF_1-2TF_2TF_1TF_2+TF_1(TF_2)^2=0
\end{aligned} \tag{17}$$

or, briefly

$$\begin{aligned}
&[H_i,H_j]=0 \qquad [H_i,E_j]=a_{ij}E_j\\
&[H_i,F_j]=-a_{ij}F_j+T^{-1}[F_j,T](H_1+H_2)\\
&\left[T^{-1}E_i,F_j\right]=\delta_{ij}\left(T^{-1}H_i+\tfrac{1}{2}(T-T^{-1})(H_1+H_2)\right)\\
&(\mathrm{ad}E_i)^{1-a_{ij}}(E_j)=0 \qquad i\neq j\\
&(\mathrm{ad}TF_i)^{1-a_{ij}}(TF_j)=0 \qquad i\neq j
\end{aligned} \tag{18}$$

where $(a_{ij})_{i,j=1,2}$ *is the Cartan matrix of* $sl(3)$.

3. We now turn to the coalgebraic structure:

Proposition 3. *The Jordanian quantum algebra* $\mathcal{U}_{\mathsf{h}}(sl(3))$ *admits a Hopf structure with coproduct, antipode and counit maps determined by*

$$\begin{aligned}
&\Delta(E_1)=E_1\otimes 1+T\otimes E_1 \qquad \Delta(E_2)=E_2\otimes 1+T\otimes E_2\\
&\Delta(T)=T\otimes T \qquad \Delta(T^{-1})=T^{-1}\otimes T^{-1}
\end{aligned}$$

$$
\begin{aligned}
\Delta(F_1) &= F_1 \otimes 1 + T^{-1} \otimes F_1 + hH_3 \otimes E_2 \\
&= F_1 \otimes 1 + T^{-1} \otimes F_1 + T(H_1 + H_2) \otimes T^{-1}[F_1, T] \\
\Delta(F_2) &= F_2 \otimes 1 + T^{-1} \otimes F_2 - hH_3 \otimes E_1 \\
&= F_2 \otimes 1 + T^{-1} \otimes F_2 + T(H_1 + H_2) \otimes T^{-1}[F_2, T] \\
\Delta(F_3) &= F_3 \otimes T + T^{-1} \otimes F_3 \\
\Delta(H_1) &= H_1 \otimes 1 + 1 \otimes H_1 - \tfrac{1}{2}(1 - T^{-2}) \otimes T^{-1} H_3 \\
&= H_1 \otimes 1 + 1 \otimes H_1 - \tfrac{1}{2}(1 - T^{-2}) \otimes (H_1 + H_2) \\
\Delta(H_2) &= H_2 \otimes 1 + 1 \otimes H_2 - \tfrac{1}{2}(1 - T^{-2}) \otimes T^{-1} H_3 \\
&= H_2 \otimes 1 + 1 \otimes H_2 - \tfrac{1}{2}(1 - T^{-2}) \otimes (H_1 + H_2) \qquad (19) \\
\Delta(H_3) &= H_3 \otimes T + T^{-1} \otimes H_3 \qquad S(E_1) = -T^{-1} E_1 \\
S(E_2) &= -T^{-1} E_2 \qquad S(T) = T^{-1} \qquad S(T^{-1}) = T \\
S(F_1) &= -TF_1 + hTH_3T^{-1}E_2 = -TF_1 + T^2(H_1 + H_2)T^{-2}[F_1, T] \\
S(F_2) &= -TF_2 - hTH_3T^{-1}E_1 = -TF_2 + T^2(H_1 + H_2)T^{-2}[F_2, T] \\
S(F_3) &= -TF_3T^{-1} \\
S(H_1) &= -H_1 - \tfrac{1}{2}(T - T^{-1})H_3 = -H_1 - \tfrac{1}{2}(T^2 - 1)(H_1 + H_2) \\
S(H_2) &= -H_2 - \tfrac{1}{2}(T - T^{-1})H_3 = -H_2 - \tfrac{1}{2}(T^2 - 1)(H_1 + H_2) \\
S(H_3) &= -TH_3T^{-1} \\
\epsilon(a) &= 0 \qquad \forall a \in \{H_1, H_2, H_3, E_1, E_2, F_1, F_2, F_3\} \\
\epsilon(T) &= \epsilon(T^{-1}) = 1.
\end{aligned}
$$

All the Hopf algebra axioms can be verified by direct calculations. We remark that our coproducts have *simpler forms* compared to those maps in [8–11]. This is one main benefit of our procedure. Pertinent to the algebraic and the coalgebraic structures of our $\mathcal{U}_h(sl(3))$ Hopf algebra described in (9), (14) and (19), here we obtain its universal $\mathcal{R}_h$-matrix in the following form:

$$\mathcal{R}_h = \exp(-hE_3 \otimes TH_3) \exp(hTH_3 \otimes E_3). \qquad (20)$$

The above universal $\mathcal{R}_h$-matrix satisfies the required properties [24] for the *full* $\mathcal{U}_h(sl(3))$ Hopf structure discussed earlier. We note that the element (20), generated by E_3 and H_3, coincides with the universal $\mathcal{R}_h$-matrix of the $\mathcal{U}_h(sl(2))$ subalgebra [25] involving the generators corresponding to the highest root, and may be connected to the results obtained by the contraction process (e.g. (5)) by a suitable twist operator that can be derived as a series expansion in h.

4. Following Drinfeld's arguments [5], it is possible to construct a twist operator $G \in \mathcal{U}(sl(3))^{\otimes 2}[[h]]$ relating the Jordanian coalgebraic structure given by (19) with the corresponding classical coalgebraic structure. For an invertible map $m : \mathcal{U}_h(sl(3)) \to \mathcal{U}(sl(3))$, $m^{-1} : \mathcal{U}(sl(3)) \to \mathcal{U}_h(sl(3))$, the following relations hold:

$$(m \otimes m) \circ \Delta \circ m^{-1}(\mathcal{X}) = G\Delta_0(\mathcal{X})G^{-1} \qquad m \circ S \circ m^{-1}(\mathcal{X}) = gS_0(\mathcal{X})g^{-1} \qquad (21)$$

where $\mathcal{X} \in \mathcal{U}(sl(3))[[h]]$ and $(\Delta_0, \epsilon_0, S_0)$ are the coproduct, counit and the antipode maps of the classical $\mathcal{U}(sl(3))$ algebra. The transforming operator $g(\in \mathcal{U}(sl(3))[[h]])$ and its inverse may be expressed as

$$g = \mu \circ (\mathrm{id} \otimes S_0)G \qquad g^{-1} = \mu \circ (S_0 \otimes \mathrm{id})G^{-1} \qquad (22)$$

where μ is the multiplication map.

For the map presented in (8), (12) and (13), we have the construction

$$G_I = 1\otimes 1 - \tfrac{1}{2}\mathsf{h}r + \tfrac{1}{8}\mathsf{h}^2\left[r^2 + 2(e_3\otimes e_3)\Delta_0(h_3)\right] - \tfrac{1}{48}\mathsf{h}^3\left[r^3 + 6(e_3\otimes e_3)\Delta_0(h_3)r - 4(\Delta_0(e_3))^2 r\right] + \tfrac{1}{384}\mathsf{h}^4\left[r^4 - 16(\Delta_0(e_3))^2r^2 + 12(e_3\otimes e_3)\Delta_0(h_3)r^2 + 12((e_3\otimes e_3)\Delta_0(h_3))^2 + 6\left({e_3}^2\otimes 1 - 1\otimes {e_3}^2\right)^2\Delta_0(h_3) + 12(\Delta_0(e_3))^2\left({e_3}^2\otimes 1 + 1\otimes {e_3}^2\right)\Delta_0(h_3) - 8\Delta_0(e_3)\left({e_3}^3\otimes 1 + 1\otimes {e_3}^3\right)\Delta_0(h_3) - 10(\Delta_0(e_3))^4\Delta_0(h_3)\right] + O(\mathsf{h}^5)$$

$$g_I = 1 + \mathsf{h}e_3\left(1+\mathsf{h}^2{e_3}^2\right)^{1/2} + \mathsf{h}^2{e_3}^2 \tag{23}$$

where the classical r-matrix reads $r = h_3\otimes e_3 - e_3\otimes h_3$. The above twist operators, while obeying the requirement (21) for the full $\mathcal{U}(sl(3))[[\mathsf{h}]]$ algebra, are, however, generated only by the elements (e_3, h_3), related to the highest root. This property accounts for the embedding of the $\mathcal{U}_\mathsf{h}(sl(2))$ algebra in the higher dimensional $\mathcal{U}_\mathsf{h}(sl(3))$ algebra. The transforming operator g_I is obtained in (23) in a closed form. The series expansion of the twist operator G_I, corresponding to the map given in (8), (12) and (13), may be developed up to an arbitrary order in h. Expansion (23) of the twist operator G_I in powers of h satisfies the cocycle condition

$$(1\otimes G_I)(\mathrm{id}\otimes\Delta_0)G_I = (G_I\otimes 1)(\Delta_0\otimes\mathrm{id})G_I \tag{24}$$

up to the desired order. Using the map given in (8), (12) and (13), the universal $\mathcal{R}_\mathsf{h}$-matrix (20) may be recast in the form

$$\mathcal{R}_\mathsf{h} = (\sigma\circ G_I)G_I^{-1} \tag{25}$$

which is valid up to an arbitrary order in expansion (23). The operator σ permutes in the tensor product space. The present discussion of the twist operator relating to the $\mathcal{U}_\mathsf{h}(sl(3))$ algebra may be easily extended to higher dimensional Jordanian algebras. A systematic study of invertible maps between the classical $\mathcal{U}(sl(2))$ and the quantum $\mathcal{U}_\mathsf{h}(sl(2))$ algebras, and the twist operators corresponding to these maps, can be found in [22]. We would like to point out here that the undeformed classical $\mathcal{U}(sl(3))$ algebra and the Jordanian $\mathcal{U}_\mathsf{h}(sl(3))$ algebra may be related by a class of maps, of which the map constructed here in (8), (12) and (13) is an example. Different maps correspond to different twist operators relating the cocommutative and the non-cocommutative coproducts of $\mathcal{U}(sl(3))$ and $\mathcal{U}_\mathsf{h}(sl(3))$ algebras, respectively. In particular, the factorized form (20) of the $\mathcal{R}_\mathsf{h}$-matrix immediately suggests the following twist operator $G_{II} = \exp(-\mathsf{h}TH_3\otimes E_3)$ in closed form. The corresponding map interrelating the classical $\mathcal{U}(sl(3))$ and the quantized $\mathcal{U}_\mathsf{h}(sl(3))$ algebras will be discussed elsewhere.

5. Let us mention that there is a $\mathbb{C}$-algebra automorphism ϕ of $\mathcal{U}_\mathsf{h}(sl(3))$ algebra such that

$$\begin{array}{lll} \phi(T^{\pm1}) = T^{\pm1} & \phi(F_3) = F_3 & \phi(H_3) = H_3 \\ \phi(E_1) = E_2 & \phi(F_1) = F_2 & \phi(H_1) = H_2 \\ \phi(E_2) = -E_1 & \phi(F_2) = -F_1 & \phi(H_2) = H_1. \end{array} \tag{26}$$

(For $\mathsf{h} = 0$, this automorphism reduces to the classical one $(h_1, e_1, f_1, h_2, e_2, f_2)\to(h_2, e_2, f_2, h_1, -e_1, -f_1)$). Also there is a second $\mathbb{C}$-algebra automorphism φ of the $\mathcal{U}_\mathsf{h}(sl(3))$ algebra defined as

$$\begin{array}{lll} \varphi(T^{\pm1}) = -T^{\pm1} & \varphi(F_3) = -F_3 & \varphi(H_3) = -H_3 \\ \varphi(E_1) = E_1 & \varphi(F_1) = F_1 & \varphi(H_1) = H_1 \\ \varphi(E_2) = E_2 & \varphi(F_2) = F_2 & \varphi(H_2) = H_2. \end{array} \tag{27}$$

6. Expressions (8), (12) and (13) permit immediate explicit construction of the finite-dimensional irreducible representations of the $\mathcal{U}_\mathsf{h}(sl(3))$ algebra. For example, the

three-dimensional irreducible representations are spanned by

$$
\begin{array}{lll}
H_1 = \begin{pmatrix} 1 & 0 & \frac{h}{2} \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -\frac{h}{2} \\ 0 & 0 & 0 \end{pmatrix} \\
H_2 = \begin{pmatrix} 0 & 0 & \frac{h}{2} \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & F_2 = \begin{pmatrix} 0 & -\frac{h}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
H_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & T^{\pm 1} = \begin{pmatrix} 1 & 0 & \pm h \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{array} \tag{28}
$$

or, by

$$
\begin{array}{lll}
H_1 = \begin{pmatrix} 1 & 0 & \frac{h}{2} \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -\frac{h}{2} \\ 0 & 0 & 0 \end{pmatrix} \\
H_2 = \begin{pmatrix} 0 & 0 & \frac{h}{2} \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & F_2 = \begin{pmatrix} 0 & -\frac{h}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
H_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & T^{\pm 1} = \begin{pmatrix} -1 & 0 & \mp h \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} .
\end{array} \tag{29}
$$

The three-irrep (29) is directly obtained from the irrep (28) using the automorphism φ.

3. $\mathcal{U}_h(sl(4))$: map and $\mathcal{R}_h$-matrix

The major interest of our approach is that it can be generalized for obtaining Jordanian quantum algebras $\mathcal{U}_h(sl(N))$ of higher dimensions. Here we illustrate our method using the $\mathcal{U}(sl(4))$ algebra as an example. Let $h_1 = e_{11} - e_{22} \equiv h_{12}$, $h_2 = e_{22} - e_{33} \equiv h_{23}$, $h_3 = e_{33} - e_{44} \equiv h_{34}$, $e_1 \equiv e_{12}$, $e_2 \equiv e_{23}$, $e_3 \equiv e_{34}$, $f_1 \equiv e_{21}$, $f_2 \equiv e_{32}$ and $f_3 \equiv e_{43}$ be the standard Chevalley generators of $\mathcal{U}(sl(4))$. The generators corresponding to other roots, obtained by the action of the Weyl group, are denoted by $e_{13} = [e_{12}, e_{23}]$, $e_{14} = [e_{13}, e_{34}]$, $e_{24} = [e_{23}, e_{34}]$, $e_{31} = [e_{32}, e_{21}]$, $e_{41} = [e_{43}, e_{31}]$, $e_{42} = [e_{43}, e_{32}]$, $h_{13} = h_{12} + h_{23}$, $h_{14} = h_{12} + h_{23} + h_{34}$ and $h_{24} = h_{23} + h_{34}$. As in the $\mathcal{U}_h(sl(3))$ algebra, the Jordanian deformation arises here from the generators corresponding to the highest root, i.e. from e_{14}, e_{41} and h_{14}. These generators are deformed as follows:

$$
\begin{aligned}
&T = he_{14} + \sqrt{1 + h^2 e_{14}^2} \qquad T^{-1} = -he_{14} + \sqrt{1 + h^2 e_{14}^2} \\
&E_{41} = e_{41} - \frac{h^2}{4} e_{14} \left(h_{14}^2 - 1\right) \qquad H_{14} = \sqrt{1 + h^2 e_{14}^2}\, h_{14}
\end{aligned} \tag{30}
$$

with the well-known coproducts

$$
\begin{aligned}
&\Delta(T) = T \otimes T \qquad \Delta(T^{-1}) = T^{-1} \otimes T^{-1} \\
&\Delta(E_{41}) = E_{41} \otimes T + T^{-1} \otimes E_{41} \\
&\Delta(H_{14}) = H_{14} \otimes T + T^{-1} \otimes H_{14}.
\end{aligned} \tag{31}
$$

Paralleling the pattern in the $\mathcal{U}_h(sl(3))$ algebra, both the subsets $\{h_{12}, e_{12}, e_{21}, e_{24}, e_{42}, h_{24} = h_{23} + h_{34}, e_{14}, e_{41}, h_{14} = h_{12} + h_{23} + h_{34}\}$ and $\{h_{13} = h_{12} + h_{23}, e_{13}, e_{31},$

$e_{34}, e_{43}, h_{34}, e_{14}, e_{41}, h_{14} = h_{12} + h_{23} + h_{34}\}$[6] are deformed exactly as presented in (12) and (13), i.e.

$$\begin{aligned}
E_{12} &= \sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{12} = T^{1/2}e_{12} \qquad E_{24} = \sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{24} = T^{1/2}e_{24} \\
E_{21} &= \sqrt{-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{21} + \frac{\mathsf{h}}{2}\sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{24}h_{14} = T^{-1/2}\left(e_{21} + \frac{\mathsf{h}}{2}Te_{24}h_{14}\right) \\
E_{42} &= \sqrt{-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{42} - \frac{\mathsf{h}}{2}\sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{12}h_{14} = T^{-1/2}\left(e_{42} - \frac{\mathsf{h}}{2}Te_{12}h_{14}\right) \\
H_{12} &= \left(-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}\right)\left(\sqrt{1+\mathsf{h}^2e_{14}^2}\, h_{12} + \frac{\mathsf{h}}{2}e_{14}(h_{12}-h_{24})\right) = h_{12} - \frac{\mathsf{h}}{2}e_{14}T^{-1}h_{14} \\
H_{24} &= \left(-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}\right)\left(\sqrt{1+\mathsf{h}^2e_{14}^2}\, h_{24} - \frac{\mathsf{h}}{2}e_{14}(h_{12}-h_{24})\right) = h_{24} - \frac{\mathsf{h}}{2}e_{14}T^{-1}h_{14}
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
E_{13} &= \sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{13} = T^{1/2}e_{13} \qquad E_{34} = \sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{34} = T^{1/2}e_{34} \\
E_{31} &= \sqrt{-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{31} + \frac{\mathsf{h}}{2}\sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{34}h_{14} = T^{-1/2}\left(e_{31} + \frac{\mathsf{h}}{2}e_{34}h_{14}\right) \\
E_{43} &= \sqrt{-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{43} - \frac{\mathsf{h}}{2}\sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}}\, e_{13}h_{14} = T^{-1/2}\left(e_{43} - \frac{\mathsf{h}}{2}e_{13}h_{14}\right) \\
H_{13} &= \left(-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}\right)\left(\sqrt{1+\mathsf{h}^2e_{14}^2}\, h_{13} + \frac{\mathsf{h}}{2}e_{14}(h_{13}-h_{34})\right) = h_{13} - \frac{\mathsf{h}}{2}e_{14}T^{-1}h_{14} \\
H_{34} &= \left(-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2e_{14}^2}\right)\left(\sqrt{1+\mathsf{h}^2e_{14}^2}\, h_{34} - \frac{\mathsf{h}}{2}e_{14}(h_{13}-h_{34})\right) = h_{34} - \frac{\mathsf{h}}{2}e_{14}T^{-1}h_{14}.
\end{aligned} \tag{33}$$

The elements E_{23}, E_{32} and H_{23} are obtained after analysing the commutators $[E_{24}, E_{43}]$ and $[E_{34}, E_{42}]$. It is simple to see that these elements remain undeformed, i.e.

$$E_{23} = e_{23} \qquad E_{32} = e_{32} \qquad H_{23} = h_{23}. \tag{34}$$

It is now easy to verify that

$$\begin{aligned}
&H_{23} + H_{34} = H_{24} \qquad [E_{12}, E_{23}] = E_{13} \qquad [E_{32}, E_{21}] = E_{31} \\
&H_{12} + H_{23} = H_{13} \qquad [E_{23}, E_{34}] = E_{24} \qquad [E_{43}, E_{32}] = E_{42}.
\end{aligned} \tag{35}$$

Proposition 4. *The generating elements* $H_1 \equiv H_{12}$, $H_2 \equiv H_{23}$, $H_3 \equiv H_{34}$, $E_1 \equiv E_{12}$, $E_2 \equiv E_{23}$, $E_3 \equiv E_{34}$, $F_1 \equiv E_{21}$, $F_2 \equiv E_{32}$, $F_3 \equiv E_{43}$ *of the Jordanian quantum algebra* $\mathcal{U}_{\mathsf{h}}(sl(4))$ *obey the following commutation rules:*

$$\begin{aligned}
&T = (1 + 2\mathsf{h}[E_1, [E_2, E_3]])^{1/2} \qquad T^{-1} = (1 + 2\mathsf{h}[E_1, [E_2, E_3]])^{-1/2} \\
&[H_1, H_2] = [H_1, H_3] = [H_2, H_3] = 0 \\
&[H_1, E_1] = 2E_1 \qquad [H_1, E_2] = -E_2 \qquad [H_1, E_3] = 0 \\
&[H_2, E_1] = -E_1 \qquad [H_2, E_2] = 2E_2 \qquad [H_2, E_3] = -E_3
\end{aligned}$$

[6] Each subset forms a $\mathcal{U}(sl(3))$ subalgebra in the $\mathcal{U}(sl(4))$ algebra.

$$
\begin{aligned}
&[H_3, E_1] = 0 \qquad [H_3, E_2] = -E_2 \qquad [H_3, E_3] = 2E_3\\
&[H_1, F_1] = -2F_1 + T^{-1}[F_1, T](H_1 + H_2 + H_3) \qquad [H_1, F_2] = F_2\\
&[H_1, F_3] = T^{-1}[F_3, T](H_1 + H_2 + H_3)\\
&[H_2, F_1] = F_1 \qquad [H_2, F_2] = -2F_2 \qquad [H_2, F_3] = F_3\\
&[H_3, F_1] = T^{-1}[F_1, T](H_1 + H_2 + H_3) \qquad [H_3, F_2] = F_2\\
&[H_3, F_3] = -2F_3 + T^{-1}[F_3, T](H_1 + H_2 + H_3)\\
&\left[T^{-1}E_1, F_1\right] = T^{-1}H_1 + \tfrac{1}{2}(T - T^{-1})(H_1 + H_2 + H_3) \qquad [E_2, F_2] = H_2\\
&\left[T^{-1}E_3, F_3\right] = T^{-1}H_3 + \tfrac{1}{2}(T - T^{-1})(H_1 + H_2 + H_3)\\
&\left[T^{-1}E_1, F_2\right] = \left[T^{-1}E_1, F_3\right] = 0 \qquad [E_2, F_1] = [E_2, F_3] = 0\\
&\left[T^{-1}E_3, F_1\right] = \left[T^{-1}E_3, F_2\right] = 0 \qquad [E_1, E_3] = [TF_1, TF_3] = 0\\
&E_1^2E_2 - 2E_1E_2E_1 + E_2E_1^2 = 0 \qquad E_1E_2^2 - 2E_2E_1E_2 + E_2^2E_1 = 0\\
&E_2^2E_3 - 2E_2E_3E_2 + E_3E_2^2 = 0 \qquad E_2E_3^2 - 2E_3E_2E_3 + E_3^2E_2 = 0\\
&(TF_1)^2F_2 - 2TF_1F_2TF_1 + F_2(TF_1)^2 = 0 \qquad TF_1F_2^2 - 2F_2TF_1F_2 + F_2^2TF_1 = 0\\
&(TF_3)^2F_2 - 2TF_3F_2TF_3 + F_2(TF_3)^2 = 0 \qquad F_2^2TF_3 - 2F_2TF_3F_2 + TF_3F_2^2 = 0
\end{aligned}
\tag{36}
$$

or, briefly,

$$
\begin{aligned}
&[H_i, H_j] = 0 \qquad [H_i, E_j] = a_{ij}E_j\\
&[H_i, F_j] = -a_{ij}F_j + (\delta_{i1} + \delta_{i3})T^{-1}[F_j, T](H_1 + H_2 + H_3)\\
&\left[T^{-(\delta_{i1}+\delta_{i3})}E_i, F_j\right] = \delta_{ij}\left(T^{-(\delta_{i1}+\delta_{i3})}H_i + \frac{(\delta_{i1} + \delta_{i3})}{2}(T - T^{-1})(H_1 + H_2 + H_3)\right)\\
&[E_i, E_j] = \left[T^{(\delta_{i1}+\delta_{i3})}F_i, T^{(\delta_{j1}+\delta_{j3})}F_j\right] = 0 \qquad |i - j| > 1\\
&(\mathrm{ad}E_i)^{1-a_{ij}}(E_j) = 0 \quad (i \neq j)\\
&\left(\mathrm{ad}T^{(\delta_{i1}+\delta_{i3})}F_i\right)^{1-a_{ij}}\left(T^{(\delta_{j1}+\delta_{j3})}F_j\right) = 0 \quad (i \neq j)
\end{aligned}
\tag{37}
$$

where $(a_{ij})_{i,j=1,2,3}$ *is the Cartan matrix of* $sl(4)$.

Proposition 5. *The non-cocommutative coproduct structure of* $\mathcal{U}_{\mathsf{h}}(sl(4))$ *reads*

$$
\begin{aligned}
&\Delta(E_1) = E_1 \otimes 1 + T \otimes E_1 \qquad \Delta(E_2) = E_2 \otimes 1 + 1 \otimes E_2\\
&\Delta(E_3) = E_3 \otimes 1 + T \otimes E_3\\
&\Delta(F_1) = F_1 \otimes 1 + T^{-1} \otimes F_1 + (H_1 + H_2 + H_3) \otimes T^{-1}[F_1, T]\\
&\Delta(F_2) = F_2 \otimes 1 + T^{-1} \otimes F_2\\
&\Delta(F_3) = F_3 \otimes 1 + T^{-1} \otimes F_3 + (H_1 + H_2 + H_3) \otimes T^{-1}[F_3, T]\\
&\Delta(H_1) = H_1 \otimes 1 + 1 \otimes H_1 - \tfrac{1}{2}(1 - T^{-2}) \otimes (H_1 + H_2 + H_3)\\
&\Delta(H_2) = H_2 \otimes 1 + 1 \otimes H_2\\
&\Delta(H_3) = H_3 \otimes 1 + 1 \otimes H_3 - \tfrac{1}{2}(1 - T^{-2}) \otimes (H_1 + H_2 + H_3).
\end{aligned}
\tag{38}
$$

Paralleling the earlier cases, the universal $\mathcal{R}_{\mathsf{h}}$-matrix of the $\mathcal{U}_{\mathsf{h}}(sl(4))$ algebra is given by

$$\mathcal{R}_{\mathsf{h}} = \exp(-\mathsf{h}E_{14} \otimes TH_{14})\exp(\mathsf{h}TH_{14} \otimes E_{14}) \tag{39}$$

where

$$E_{14} = \mathsf{h}^{-1}\ln T = \mathsf{h}^{-1}\mathrm{arcsinh}\,(\mathsf{h}e_{14}). \tag{40}$$

4. $\mathcal{U}_{\mathsf{h}}(sl(N))$: generalization

The $\mathcal{U}_{\mathsf{h}}(sl(5))$ algebra is derived in a similar way: The elements E_2, E_3, F_2, F_3, H_2, H_3 are not affected by the non-standard quantization. From the above studies, it is easy to see that

Proposition 6. *The Chevalley generators* $(E_i, F_i, H_i \mid i = (1, \dots, N-1))$ *of the Jordanian deformed* $\mathcal{U}_{\mathsf{h}}(sl(N))$ *algebra may be mapped on the classical* $sl(N)$ *algebra as follows:*

$$\begin{aligned}
T &= \mathsf{h}[e_1, [e_2, \dots, [e_{N-2}, e_{N-1}] \cdots]] + \sqrt{1 + \mathsf{h}^2([e_1, [e_2, \dots, [e_{N-2}, e_{N-1}] \cdots]])^2} \\
T^{-1} &= -\mathsf{h}[e_1, [e_2, \dots, [e_{N-2}, e_{N-1}] \cdots]] + \sqrt{1 + \mathsf{h}^2([e_1, [e_2, \dots, [e_{N-2}, e_{N-1}] \cdots]])^2} \\
E_i &= T^{(\delta_{i1}+\delta_{i,N-1})/2} e_i \\
F_i &= T^{-(\delta_{i1}+\delta_{i,N-1})/2} \left(f_i + \frac{\mathsf{h}}{2} T[f_i, [e_1, [e_2, \dots, [e_{N-2}, e_{N-1}] \cdots]]](h_1 + \cdots + h_{N-1}) \right) \\
H_i &= h_i - \frac{(\delta_{i1} + \delta_{i,N-1})\mathsf{h}}{2} [e_1, [e_2, \dots, [e_{N-2}, e_{N-1}] \cdots]] T^{-1}(h_1 + \cdots + h_{N-1})
\end{aligned} \tag{41}$$

and they satisfy the commutation relations

$$\begin{aligned}
&[H_i, H_j] = 0 \qquad [H_i, E_j] = a_{ij} E_j \\
&[H_i, F_j] = -a_{ij} F_j + (\delta_{i1} + \delta_{i,N-1}) T^{-1} [F_j, T](H_1 + \cdots + H_{N-1}) \\
&\left[T^{-(\delta_{i1}+\delta_{i,N-1})} E_i, F_j\right] = \delta_{ij} \left(T^{-(\delta_{i1}+\delta_{i,N-1})} H_i + \frac{(\delta_{i1} + \delta_{i,N-1})}{2} (T - T^{-1})(H_1 + \cdots + H_{N-1}) \right) \\
&[E_i, E_j] = 0 \qquad |i - j| > 1 \\
&\left[T^{(\delta_{i1}+\delta_{i,N-1})} F_i, T^{(\delta_{j1}+\delta_{j,N-1})} F_j\right] = 0 \qquad |i - j| > 1 \\
&(\mathrm{ad} E_i)^{1-a_{ij}} (E_j) = 0 \qquad (i \neq j) \\
&\left(\mathrm{ad} T^{(\delta_{i1}+\delta_{i,N-1})} F_i\right)^{1-a_{ij}} \left(T^{(\delta_{j1}+\delta_{j,N-1})} F_j\right) = 0 \qquad (i \neq j)
\end{aligned} \tag{42}$$

where $(a_{ij})_{i,j=1,\dots,N}$ *is the Cartan matrix of* $sl(N)$, *i.e.* $a_{ii} = 2$, $a_{i,i\pm1} = -1$ *and* $a_{ij} = 0$ *for* $|i - j| > 1$.

The algebra (42) is called the *Jordanian quantum algebra* $\mathcal{U}_{\mathsf{h}}(sl(N))$. Expressions (41) may be regarded as a particular nonlinear realization of the $\mathcal{U}_{\mathsf{h}}(sl(N))$ generators. Other nonlinear realizations of the $\mathcal{U}_h(sl(N))$ algebra in terms of the classical $sl(N)$ generators may also be obtained.

Proposition 7. *The Jordanian* $\mathcal{U}_{\mathsf{h}}(sl(N))$ *algebra (42) admits the following coalgebra structure:*

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes 1 + T^{(\delta_{i1}+\delta_{i,N-1})} \otimes E_i \\
\Delta(F_i) &= F_i \otimes 1 + T^{-(\delta_{i1}+\delta_{i,N-1})} \otimes F_i + T(H_1 + \cdots + H_{N-1}) \otimes T^{-1}[F_i, T] \\
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i - \frac{(\delta_{i1} + \delta_{i,N-1})}{2} (1 - T^{-2}) \otimes (H_1 + \cdots + H_{N-1}) \\
S(E_i) &= -T^{-(\delta_{i1}+\delta_{i,N-1})} E_i \\
S(F_i) &= -T^{(\delta_{i1}+\delta_{i,N-1})} F_i + T^2 (H_1 + \cdots + H_{N-1}) T^{-2} [F_i, T] \\
S(H_i) &= -H_i + \frac{(\delta_{i1} + \delta_{i,N-1})}{2} (1 - T^2)(H_1 + \cdots + H_{N-1}) \\
\epsilon(E_i) &= \epsilon(F_i) = \epsilon(H_i) = 0.
\end{aligned} \tag{43}$$

Following (20) and (39), we obtain the universal $\mathcal{R}_{\mathsf{h}}$-matrix of an arbitrary $\mathcal{U}_{\mathsf{h}}(sl(N))$ algebra in the following general form:

$$\mathcal{R}_{\mathsf{h}} = \exp(-\mathsf{h} E_{1N} \otimes T H_{1N}) \exp(\mathsf{h} T H_{1N} \otimes E_{1N}). \tag{44}$$

where

$$H_{1N} = T(H_1 + \cdots + H_{N-1}) \qquad E_{1N} = \mathsf{h}^{-1} \ln T = \mathsf{h}^{-1} \text{arcsinh}\,(\mathsf{h} e_{1N}). \tag{45}$$

The above universal $\mathcal{R}_{\mathsf{h}}$-matrix of the *full* $\mathcal{U}_{\mathsf{h}}(sl(N))$ Hopf algebra is obtained from the generators associated with the highest root; and its form coincides with the universal $\mathcal{R}_{\mathsf{h}}$-matrix of the $\mathcal{U}_{\mathsf{h}}(sl(2))$ Hopf subalgebra [25] associated with the highest root. It is interesting to note that the nonlinear map (41) equips the h-deformed generators (E_i, F_i, H_i) with an additional induced co-commutative coproduct. Similarly, the undeformed generators (e_i, f_i, h_i), via the inverse map, may be viewed as elements of the $\mathcal{U}_{\mathsf{h}}(sl(N))$ algebra; and, thus, may be endowed with an induced non-cocommutative coproduct.

5. Conclusion

In general, a class of nonlinear invertible maps exists relating the Jordanian quantum algebras and their classical analogues. Here we have used a particular map realizing Jordanian $\mathcal{U}_{\mathsf{h}}(sl(N))$ algebra for an arbitrary *N*. As a result of our choice of the basis, via the map described earlier, the algebraic commutation relations are *deformed*. One benefit of the method is that the Ohn's $\mathcal{U}_{\mathsf{h}}(sl(2))$ algebra is embedded as a Hopf subalgebra in our construction of the $\mathcal{U}_{\mathsf{h}}(sl(N))$ Hopf structure. Another important advantage of our procedure is that *our expressions for the coalgebraic structure are considerably simpler than those found elsewhere* [8–11]. For our choice of the Hopf structure of the $\mathcal{U}_{\mathsf{h}}(sl(N))$ algebra, we obtain its universal $\mathcal{R}_{\mathsf{h}}$-matrix expressed in terms of the generators corresponding to the highest root. The twist operator corresponding to our map and relating the classical cocommutative with the Jordanian non-cocommutative Hopf structures has been obtained as a series expansion in the deformation parameter h.

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